

EMBEDDINGS OF PROPER METRIC SPACES INTO BANACH SPACES

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ABSTRACT. We show that there exists a strong uniform embedding from any proper metric space into any Banach space without cotype. Then we prove a result concerning the Lipschitz embedding of locally finite subsets of \mathcal{L}_p -spaces. We use this locally finite result to construct a coarse bi-Lipschitz embedding for proper subsets of any \mathcal{L}_p -space into any Banach space X containing the ℓ_p^n 's. Finally using an argument of G. Schechtman we prove that for general proper metric spaces and for Banach spaces without cotype a converse statement holds.

1. INTRODUCTION AND NOTATION

We fix some notation and give some motivation concerning the embedding of metric spaces.

Let (M, d) and (N, δ) be two metric spaces and $f : M \rightarrow N$ be a map. For $t > 0$ define :

$$\rho_f(t) = \inf\{\delta(f(x), f(y)), \quad d(x, y) \geq t\}$$

and

$$\omega_f(t) = \sup\{\delta(f(x), f(y)), \quad d(x, y) \leq t\}.$$

We say that f is a *coarse embedding* if $\omega_f(t)$ is finite for all $t > 0$ and $\lim_{t \rightarrow \infty} \rho_f(t) = \infty$.

Suppose now that f is injective. We say that f is a *uniform embedding* if f and f^{-1} are uniformly continuous (i.e $\omega_f(t)$ and $\omega_{f^{-1}}(t)$ tend to 0 when t tends to 0). According to [12] we refer to f as a *strong uniform embedding* if it is simultaneously a uniform embedding and a coarse embedding.

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Following [18], we also define the *distortion* of f to be

$$\text{dist}(f) := \|f\|_{Lip} \|f^{-1}\|_{Lip} = \sup_{x \neq y \in M} \frac{\delta(f(x), f(y))}{d(x, y)} \cdot \sup_{x \neq y \in M} \frac{d(x, y)}{\delta(f(x), f(y))}.$$

If the distortion of f is finite, we say that f is a *Lipschitz embedding* and that M Lipschitz embeds into N . The terminology *metric embedding* is used too.

In the sequel our terminology for classical notions in Banach space theory follows [16] and [2]. Let X and Y be two Banach spaces. If X and Y are linearly isomorphic, the *Banach-Mazur distance* between X and Y , denoted by $d_{BM}(X, Y)$, is the infimum of $\|T\| \|T^{-1}\|$, over all linear isomorphisms T from X onto Y .

For $p \in [1, \infty]$ and $n \in \mathbb{N}$, ℓ_p^n denotes the space \mathbb{R}^n equipped with the ℓ_p norm. We say that a Banach space X uniformly contains the ℓ_p^n 's if there is a constant $C \geq 1$ such that for every integer n , X admits an n -dimensional subspace Y so that $d_{BM}(\ell_p^n, Y) \leq C$.

The problem of embedding of metric spaces occurs very often in different fields of mathematics. Let us focus on the following two. The first one deals with Lipschitz classification of Banach spaces and is related to the celebrated result of I. Aharoni [1] which asserts that every separable metric space Lipschitz embeds into c_0 (the Banach space of all real sequences converging to 0, equipped with the supremum norm). Is the converse statement of I. Aharoni's result true? Namely, if a Banach space which is Lipschitz universal for the class of all separable metric spaces does it necessarily contain a isomorphic copy of c_0 ? This question is equivalent to the following problem :

Problem I : If c_0 Lipschitz embeds into a Banach space X does it imply that c_0 is isomorphic to a closed subspace of X ?

This question is difficult but a positive result for the isometric case can be found in [10] where G. Godefroy and N. J. Kalton showed, using Lipschitz-free Banach spaces, that if a separable Banach space is isometric to a subset of another Banach space Y , then it is actually linearly isometric to a subspace of Y . There exist counterexamples in the nonseparable case. Very recently, in [9] Y. Dutrieux and G. Lancien prove that a Banach space containing isometrically every compact metric space must contain a subspace linearly isometric to $C([0, 1])$ (the space of all continuous functions on $[0, 1]$ equipped with the supremum norm). In the same vein, in [13] N. J. Kalton proved that if c_0 coarsely or uniformly embeds into a Banach space X then one of its iterated duals has to be nonseparable (in particular, X cannot be reflexive). On the other hand, N. J. Kalton proved in [12] that c_0 can be strongly uniformly embedded into a Banach space with the Schur property. Thus

c_0 can be uniformly and coarsely embedded into a Banach space which does not contain a linear copy of c_0 .

The generalization of Index Theory for non-compact Riemannian manifolds gives birth to the second problem.

Problem II: Which metric spaces can be coarsely embedded into a super-reflexive or reflexive Banach space?

The strong result of Kasparov and Yu in [15] gives a precise idea of this connection.

They showed that if M is a discrete metric space with bounded geometry and if M is coarsely embeddable into a uniformly convex Banach space, then the coarse geometric Novikov conjecture holds for M .

The definition of a metric space with bounded geometry is the following :

A metric space M is *locally finite* if any ball of M with finite radius is finite. If moreover, there is a function $C : (0, +\infty) \rightarrow \mathbb{N}$ such that any ball of radius r contains at most $C(r)$ points, we say that M has a *bounded geometry*.

Problem II received an upsurge of interest in the light of Kasparov and Yu's result. Some significant partial answers have been given in [7] and [5] where it is proved that any locally finite metric space Lipschitz embeds into any Banach space without cotype. We recall that a metric space is *stable* if for any pair of sequences $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty$ such that both iterated limits exist we have

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} d(x_m, y_n) = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} d(x_m, y_n).$$

In [13] it is proved that a stable metric space can always be coarsely or uniformly embedded into a reflexive Banach space. In the next section we improve this result for proper metric spaces, which are automatically stable.

2. STRONG UNIFORM EMBEDDING OF PROPER METRIC SPACES

A metric space is proper if all its closed balls are compact. In this section we will prove the following theorem :

Theorem 2.1. *Let (M, d) a proper metric space and X a Banach space without cotype, then M strongly uniformly embeds into X .*

Proof : Let X be a Banach space uniformly containing the ℓ_∞^n 's (or equivalently without any nontrivial cotype).

Fix $t_0 \in M$, a proper metric space, and denote for $n \in \mathbb{Z}$, $B_n = B(t_0, 2^{n+1})$ the closed ball of M of radius 2^{n+1} centered at the point t_0 .

Let M_n^k a maximal 2^{-k+n+3} -net of B_n , with $k \geq 1$.

We define

$$\varphi_n^k : B_n \rightarrow \ell_\infty(M_n^k) := X_{n,k}$$

$$t \mapsto (d(t, s) - |s|)_{s \in M_n^k}, \text{ where } |s| = d(s, t_0).$$

Let $\Phi: I = \mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{Z}^+$ a bijection. Following [5] we can build inductively finite dimensional subspaces $(F_j)_{j=0}^\infty$ of X and $(T_j)_{j=0}^\infty$ so that for every $j \geq 0$, T_j is a linear isomorphism from $X_{\Phi^{-1}(j)}$ onto F_j satisfying

$$\forall u \in X_{\Phi^{-1}(j)} \quad \frac{1}{2}\|u\| \leq \|T_j u\| \leq \|u\|$$

and also such that $(F_j)_{j=0}^\infty$ is a Schauder finite dimensional decomposition of its closed linear span Z . Let P_j be the projection from Z onto $F_0 \oplus \dots \oplus F_j$ with kernel $\overline{\text{Span}}(\bigcup_{i=j+1}^\infty F_i)$, we may assume after renorming, that $\|P_j\| \leq 1$. Finally we denote $\Pi_0 = P_0$ and $\Pi_j = P_j - P_{j-1}$ for $j \geq 1$. We have that $\|\Pi_j\| \leq 2$.

Define now

$$f_n^k : B_n \rightarrow F_{\Phi(n,k)}$$

$$t \mapsto T_{\Phi(n,k)}(\varphi_n^k(t))$$

and,

$$f_n : B_n \rightarrow \sum_{k \geq 1} \Pi_{\Phi(n,k)}(Z)$$

$$t \mapsto \sum_{k \geq 1} \frac{f_n^k(t)}{(n-k)^2 + 1}$$

It is obvious that f_n is C -Lipschitz with $C = \sum_{n \in \mathbb{Z}} \frac{1}{n^2 + 1}$.

Finally the embedding is

$$f : M \rightarrow Z = \overline{\text{Span}}(\bigcup_{i=0}^\infty F_i) \subset X$$

$$t \mapsto \lambda_t f_n(t) + (1 - \lambda_t) f_{n+1}(t), \text{ if } 2^n \leq |t| \leq 2^{n+1}$$

where,

$$\lambda_t = \frac{2^{n+1} - |t|}{2^n}, \quad n \in \mathbb{Z}$$

We start by showing that f is Lipschitz. Let $a, b \in M$ and assume, as we may, that $|a| \leq |b|$.

I) If $|a| \leq \frac{1}{2}|b|$, then

$$\|f(a) - f(b)\| \leq C(|a| + |b|) \leq \frac{3C}{2}|b| \leq 3C(|b| - |a|) \leq 3Cd(a, b).$$

II) If $\frac{1}{2}|b| < |a| \leq |b|$, we have two different cases to consider.

1) $2^n \leq |a| \leq |b| < 2^{n+1}$, for some n . Then, let

$$\lambda_a = \frac{2^{n+1} - |a|}{2^n} \quad \text{and} \quad \lambda_b = \frac{2^{n+1} - |b|}{2^n}.$$

We have that

$$|\lambda_a - \lambda_b| = \frac{|b| - |a|}{2^n} \leq \frac{d(a, b)}{2^n},$$

so

$$\begin{aligned} \|f(a) - f(b)\| &= \|\lambda_a f_n(a) - \lambda_b f_n(b) + (1 - \lambda_a)f_{n+1}(a) \\ &\quad - (1 - \lambda_b)f_{n+1}(b)\| \\ &\leq \lambda_a \|f_n(a) - f_n(b)\| + (1 - \lambda_a)\|f_{n+1}(a) - f_{n+1}(b)\| \\ &\quad + 2C|\lambda_a - \lambda_b||b| \\ &\leq Cd(a, b) + 2^{n+2}C|\lambda_a - \lambda_b| \\ &\leq 5Cd(a, b). \end{aligned}$$

2) $2^n \leq |a| < 2^{n+1} \leq |b| < 2^{n+2}$, for some n . Then, let

$$\lambda_a = \frac{2^{n+1} - |a|}{2^n} \quad \text{and} \quad \lambda_b = \frac{2^{n+2} - |b|}{2^{n+1}}.$$

We have that,

$$\lambda_a \leq \frac{d(a, b)}{2^n}, \quad \text{so} \quad \lambda_a |a| \leq 2d(a, b).$$

Similarly,

$$1 - \lambda_b = \frac{|b| - 2^{n+1}}{2^{n+1}} \leq \frac{d(a, b)}{2^{n+1}} \quad \text{and} \quad (1 - \lambda_b)|b| \leq 2d(a, b).$$

$$\begin{aligned}
\|f(a) - f(b)\| &= \|\lambda_a f_n(a) + (1 - \lambda_a) f_{n+1}(a) - \lambda_b f_{n+1}(b) \\
&\quad - (1 - \lambda_b) f_{n+2}(b)\| \\
&\leq \lambda_a (\|f_n(a)\| + \|f_{n+1}(a)\|) + (1 - \lambda_b) (\|f_{n+1}(b)\| \\
&\quad + \|f_{n+2}(b)\|) + \|f_{n+1}(a) - f_{n+1}(b)\| \\
&\leq 2C\lambda_a |a| + 2C(1 - \lambda_b) |b| + 2Cd(a, b) \\
&\leq 9C d(a, b).
\end{aligned}$$

We have shown that f is $9C$ -Lipschitz.

We shall now estimate f from below. We consider $a, b \in M$ and assume again that $|a| \leq |b|$. We need to study three different cases. In our discussion, whenever $|a|$ (respectively $|b|$) will belong to $[2^m, 2^{m+1})$, for some integer m , we shall denote

$$\lambda_a = \frac{2^{m+1} - |a|}{2^m} \quad (\text{respectively } \lambda_b = \frac{2^{m+1} - |b|}{2^m}).$$

In the sequel we denote $R_{(n,k)} = T_{\Phi_{(n,k)}}^{-1} \circ \Pi_{\Phi_{(n,k)}}$ for the sake of convenience.

1) $2^n \leq |a| \leq |b| < 2^{n+1}$, for some n .

There exists $l \geq 1$ and $s_b \in M_n^{l+3}$ such that $2^{-l+n+2} \leq d(a, b) < 2^{-l+n+3}$ and $d(s_b, b) < 2^{-l+n}$. We denote $k = l + 3$ in the sequel.

Thus, $R_{(n,k)}(f(a) - f(b)) = \frac{1}{(n-l-3)^2+1} (\lambda_a \varphi_n^k(a) - \lambda_b \varphi_n^k(b))$ and so,

$$\begin{aligned}
\lambda_a \varphi_n^k(a) - \lambda_b \varphi_n^k(b) &= (\lambda_a (d(a, s) - |s|) - \lambda_b (d(b, s) - |s|))_{s \in M_n^k} \\
(\lambda_a \varphi_n^k(a) - \lambda_b \varphi_n^k(b))_{|s_b} &= \lambda_a d(a, s_b) + (\lambda_b - \lambda_a) |s_b| - \lambda_b d(b, s_b)
\end{aligned}$$

On the other hand,

$$R_{(n+1,k)}(f(a) - f(b)) = \frac{1}{(n-l-2)^2+1} ((1 - \lambda_a) \varphi_{n+1}^k(a) - (1 - \lambda_b) \varphi_{n+1}^k(b))$$

and,

$$\begin{aligned}
(1 - \lambda_a) \varphi_{n+1}^k(a) - (1 - \lambda_b) \varphi_{n+1}^k(b) &= ((1 - \lambda_a) (d(a, s) - |s|) \\
&\quad - (1 - \lambda_b) (d(b, s) - |s|))_{s \in M_{n+1}^k} \\
((1 - \lambda_a) \varphi_{n+1}^k(a) - (1 - \lambda_b) \varphi_{n+1}^k(b))_{|s_b} &= (1 - \lambda_a) d(a, s_b) + (\lambda_a - \lambda_b) |s_b| \\
&\quad - (1 - \lambda_b) d(b, s_b)
\end{aligned}$$

We remark that

$$n - l - 3 \leq n - l - 2 \leq n - l + 2 \leq \log(d(a, b)) < n - l + 3$$

thus if $0 \leq n - l - 3$ then

$$(n - l - 3)^2 + 1 \leq (n - l + 2)^2 + 1 \leq (\log d(a, b))^2 + 1$$

otherwise

$$\log(d(a, b)) - 7 \leq n - l - 3 \leq n - l - 2 < 0$$

and we have

$$(n - l - 2)^2 + 1 \leq (n - l - 3)^2 + 1 \leq \left(\log \frac{d(a, b)}{2^7} \right)^2 + 1.$$

Let $\alpha(t) = (\log(t))^2 + 1$. Now if we sum $R_{(n+1,k)}(f(a) - f(b))$ and $R_{(n,k)}(f(a) - f(b))$ we get

$$\begin{aligned} 8 \cdot \text{Max} \left\{ \alpha(d(a, b)), \alpha \left(\frac{d(a, b)}{2^7} \right) \right\} \|f(a) - f(b)\| &\geq d(a, s_b) - d(b, s_b) \\ &\geq d(a, b) - 2d(b, s_b) \\ &\geq d(a, b) - 2^{-l+n+1} \\ &\geq \frac{d(a, b)}{2} \end{aligned}$$

2) $2^n \leq |a| < 2^{n+1} \leq |b| < 2^{n+2}$, for some n .

There exists $l \geq 1$ and $s_a \in M_{n+1}^k$ such that $2^{-l+n+3} \leq d(a, b) < 2^{-l+n+4}$ and $d(s_a, a) < 2^{-l+n}$.

We use again the projections and we evaluate at some specific points.

$$\begin{aligned} R_{(n,k)}(f(a) - f(b))|_{t_0} &= \frac{1}{(n-k)^2+1} (\lambda_a d(t, s) - \lambda_a |s|)|_{s=t_0} \\ &= \frac{\lambda_a}{(n-k)^2+1} |a| \end{aligned}$$

$$\begin{aligned} R_{(n+2,k)}(f(a) - f(b))|_{t_0} &= \frac{1}{(n-k+2)^2+1} (-(1 - \lambda_b)d(b, s) \\ &\quad -(1 - \lambda_b)|s|)|_{s=t_0} \\ &= -\frac{(1-\lambda_b)}{(n-k+2)^2+1} |b| \end{aligned}$$

$$\begin{aligned} R_{(n+1,k)}(f(a) - f(b))|_{s_a} &= \frac{1}{(n-k+1)^2+1} ((1 - \lambda_a)d(a, s) - (1 - \lambda_a)|s| \\ &\quad - \lambda_b d(b, s) + \lambda_b |s|)|_{s=s_a} \\ &= \frac{1}{(n-k+1)^2+1} ((1 - \lambda_a)d(a, s_a) - \lambda_b d(b, s_a) \\ &\quad - (1 - \lambda_a)|s_a| + \lambda_b |s_a|) \end{aligned}$$

Thus,

$$\begin{aligned}
12 \text{Max} \left\{ \alpha(d(a, b)), \alpha \left(\frac{d(a, b)}{2^7} \right) \right\} \|f(a) - f(b)\| &\geq \lambda_a |a| + (1 - \lambda_b) |b| \\
&\quad - (1 - \lambda_a) d(a, s_a) + \lambda_b d(b, s_a) \\
&\quad + (1 - \lambda_a) |s_a| - \lambda_b |s_a| \\
&\geq \lambda_b d(a, b) + \lambda_a |a| + (1 - \lambda_b) |b| \\
&\quad + (1 - \lambda_a) |a| - \lambda_b |a| \\
&\quad - \lambda_b d(s_a, a) - (1 - \lambda_a) d(s_a, a) \\
&\quad - (1 - \lambda_a) d(a, s_a) - \lambda_b d(s_a, a) \\
&\geq d(a, b) - 2(1 - \lambda_a + \lambda_b) d(a, s_a) \\
&\geq d(a, b) - 4d(s_a, a) \\
&\geq \frac{d(a, b)}{2}
\end{aligned}$$

3) $2^n \leq |a| < 2^{n+1} < 2^p \leq |b| < 2^{p+1}$ for some integers n and p .
There exists $l \geq 1$ such that $2^{-l+p+2} \leq d(a, b) < 2^{-l+p+3}$

$$\begin{aligned}
R_{(p,l)}(f(a) - f(b))|_{t_0} &= \frac{1}{(p-l)^2+1} (-\lambda_b d(b, s) + \lambda_b |s|)|_{s=t_0} \\
&= -\frac{\lambda_b}{(p-l)^2+1} |b| \\
R_{(p+1,l)}(f(a) - f(b))|_{t_0} &= \frac{1}{(p-l+1)^2+1} (-(1 - \lambda_b) d(b, s) \\
&\quad + (1 - \lambda_b) |s|)|_{s=t_0} \\
&= -\frac{(1-\lambda_b)}{(p-l+1)^2+1} |b|
\end{aligned}$$

$$\begin{aligned}
8 \cdot \text{Max} \left\{ \alpha(d(a, b)), \alpha \left(\frac{d(a, b)}{2^7} \right) \right\} \|f(a) - f(b)\| &\geq \lambda_b |b| + (1 - \lambda_b) |b| \\
&\geq |b| \\
&\geq \frac{2}{3} (|a| + |b|) \\
&\geq \frac{2}{3} d(a, b)
\end{aligned}$$

All possible cases are settled and we have shown that f is a strong uniform embedding which satisfies the following estimates:

$$\gamma(d(a, b)) \leq \|f(a) - f(b)\| \leq 9Cd(a, b),$$

where

$$\gamma(t) = \frac{t}{24 \cdot \text{Max} \left\{ \alpha(t), \alpha\left(\frac{t}{2^7}\right) \right\}}$$

has the following behavior

$$\begin{aligned} \gamma(t) &= \frac{t}{(\log(t))^2 + 1} \text{ if } t > 2^7 \\ &\geq D \text{ a positive constant if } 1 \leq t \leq 2^7 \\ &= \frac{t}{\left(\log\left(\frac{t}{2^7}\right)\right)^2 + 1} \text{ if } t < 1. \end{aligned}$$

□

3. LIPSCHITZ EMBEDDING FOR LOCALLY FINITE SUBSETS OF \mathcal{L}_p -SPACES

Let $1 \leq p \leq \infty$ and $\lambda \geq 1$. A Banach space X is a $\mathcal{L}_{p,\lambda}$ -space if for every finite-dimensional subspace F of X there exists a finite-dimensional subspace $G \subset X$ containing F such that $d_{BM}(G, \ell_p^m) \leq \lambda$ (where m is the dimension of G). X is a \mathcal{L}_p -space if it is a $\mathcal{L}_{p,\lambda}$ -space for some λ . We refer to [6] Appendix F for results on \mathcal{L}_p -spaces. Several results on embeddings of locally finite sets were obtained by M. I. Ostrovskii in [20], the author and G. Lancien in [5].

Theorem 3.1. *[Baudier-Lancien] Let X be a Banach space without cotype and let (M, d) be a locally finite metric space. Then there exists a Lipschitz embedding of M into X .*

Theorem 3.2. *[Ostrovskii] Let (M, d) be a locally finite subset of a Hilbert space. Then M is Lipschitz embeddable into an arbitrary infinite dimensional Banach space.*

We prove a similar result in the broad context of \mathcal{L}_p -spaces which extends theorems 3.1 and 3.2.

Proposition 3.3. *Let $1 \leq p \leq \infty$ and $\lambda \geq 1$. Let $(Y, \|\cdot\|)$ a $\mathcal{L}_{p,\lambda}$ -space and M a locally finite subset of Y , then M Lipschitz embeds into any Banach space X which contains uniformly the ℓ_p^n 's. Moreover the distortion is universal.*

Proof : Let $B_n := \{t \in M; \|t\| \leq 2^{n+1}\}$ for $n \in \mathbb{N}$. We may assume that $B(t_0, 1) = \{t_0\}$ and that $t_0 = 0$. $F_n = \overline{\text{Span} B_n}$ is a finite-dimensional subspace of Y , hence there exists a finite-dimensional subspace G_n containing F_n and an isomorphism R_n from G_n onto $\ell_p^{\dim(G_n)}$ such that $\text{dist}(R_n) \leq \lambda$, with $\|R_n\| \leq 1$ and $\|R_n^{-1}\| \leq \lambda$. Fix $\delta > 0$. Like in the

proof of theorem 2.1 we build inductively a Schauder finite-dimensional decomposition $(Z_n)_{n=0}^\infty$ of a subspace of X and $(T_n)_{n=0}^\infty$ so that for every $n \geq 0$, T_n is a linear isomorphism from $\ell_p^{dim(G_n)}$ onto Z_n satisfying

$$\forall u \in \ell_p^{dim(G_n)} \quad \frac{1}{1+\delta} \|u\| \leq \|T_n u\| \leq \|u\|.$$

We keep the same notations for the different projections.

We define

$$\begin{aligned} f_n : G_n \supset B_n &\longrightarrow Z \subset X \\ t &\longmapsto T_n \circ R_n(t) \end{aligned}$$

And then we construct $f : M \longrightarrow Z \subset X$ as follows:

- (i) $f(0) = 0$.
- (ii) For $n \geq 0$ and $2^n \leq \|t\| < 2^{n+1}$:

$$f(t) = \lambda_t f_n(t) + (1 - \lambda_t) f_{n+1}(t), \quad \text{where } \lambda_t = \frac{2^{n+1} - \|t\|}{2^n}.$$

In the sequel we will always suppose that $\|a\| \leq \|b\|$. We have to consider three cases.

- (1) $2^n \leq \|a\| \leq \|b\| < 2^{n+1}$

We have

$$(3.1) \quad \|\Pi_n(f(a) - f(b))\| = \|\lambda_a f_n(a) - \lambda_b f_n(b)\|$$

and,

$$(3.2) \quad \|\Pi_{n+1}(f(a) - f(b))\| = \|(1 - \lambda_a) f_{n+1}(a) - (1 - \lambda_b) f_{n+1}(b)\|$$

By the triangle inequality (3.1) is between

$$(3.3) \quad \lambda_a \|f_n(a) - f_n(b)\| \stackrel{+}{-} (\lambda_a - \lambda_b) \|f_n(b)\|$$

And (3.2) between

$$(3.4) \quad (1 - \lambda_a) \|f_{n+1}(a) - f_{n+1}(b)\| \stackrel{+}{-} (\lambda_a - \lambda_b) \|f_{n+1}(b)\|$$

So we get

$$\frac{1}{4} \left(\frac{\|a - b\|}{\lambda(1 + \delta)} - 2(\lambda_a - \lambda_b) \|b\| \right) \leq \|f(a) - f(b)\| \leq \|a - b\| + 2(\lambda_a - \lambda_b) \|b\|$$

But $(\lambda_a - \lambda_b) \|b\| \leq \frac{\|b\| - \|a\|}{2^n} 2^{n+1}$, hence if $\|b\| - \|a\| \leq \frac{\|a - b\|}{5(1 + \delta)\lambda}$ we have,

$$\frac{\|a - b\|}{20(1 + \delta)\lambda} \leq \|f(a) - f(b)\| \leq 5\|a - b\|$$

If $\|b\| - \|a\| \geq \frac{\|a - b\|}{5(1 + \delta)\lambda}$, using the linearity we get,

$$\begin{aligned}
 4\|f(a) - f(b)\| &\geq \frac{\|\lambda_a a - \lambda_b b\|}{\lambda(1+\delta)} + \frac{\|(1-\lambda_a)a - (1-\lambda_b)b\|}{\lambda(1+\delta)} \\
 &\geq \frac{1}{\lambda(1+\delta)} (\lambda_b \|b\| - \lambda_a \|a\| + (1-\lambda_b)\|b\| - (1-\lambda_a)\|a\|) \\
 &\geq \frac{\|a-b\|}{5(1+\delta)^2 \lambda^2}
 \end{aligned}$$

$$(2) \quad 2^n \leq \|a\| < 2^{n+1} \leq \|b\| < 2^{n+2}$$

The quantity $\|f(a) - f(b)\|$ is between the sum and the half-max of the numbers

$$(3.5) \quad \|\Pi_n(f(a) - f(b))\| = \|\lambda_a f_n(a)\|,$$

$$(3.6) \quad \|\Pi_{n+1}(f(a) - f(b))\| = \|(1-\lambda_a)f_{n+1}(a) - \lambda_b f_{n+1}(b)\| \text{ and}$$

$$(3.7) \quad \|\Pi_{n+2}(f(a) - f(b))\| = \|(1-\lambda_b)f_{n+2}(b)\|.$$

But,

$$(3.5) \leq \frac{2^{n+1} - \|a\|}{2^n} \|a\| \leq 2(2^{n+1} - \|a\|) \leq 2(\|b\| - \|a\|) \leq 2\|b - a\|.$$

$$(3.7) \leq \frac{\|b\| - 2^{n+1}}{2^{n+1}} \|b\| \leq 2\|b - a\|.$$

$$\begin{aligned}
 (3.6) &= \|f_{n+1}(a) - f_{n+1}(b) + (1-\lambda_b)f_{n+1}(b) - \lambda_a f_{n+1}(a)\| \\
 &\leq \|a - b\| + 2\|a - b\| + 2\|a - b\| \\
 &\leq 5\|a - b\|.
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 (3.5) &\geq \frac{\lambda_a \|a\|}{\lambda(1+\delta)} \\
 &\geq \frac{1}{\lambda(1+\delta)} \frac{2^{n+1} - \|a\|}{2^n} \|a\| \\
 &\geq \frac{2^{n+1} - \|a\|}{\lambda(1+\delta)}
 \end{aligned}$$

And, $(3.7) \geq \frac{(1-\lambda_b)\|b\|}{\lambda(1+\delta)} \geq \frac{\|b\| - 2^{n+1}}{\lambda(1+\delta)}$, hence

$$2\|f(a) - f(b)\| \geq \text{Max} \left\{ \frac{2^{n+1} - \|a\|}{\lambda(1+\delta)}, \frac{\|b\| - 2^{n+1}}{\lambda(1+\delta)} \right\}.$$

If $\text{Max}\{2^{n+1} - \|a\|, \|b\| - 2^{n+1}\} \geq \frac{\|b-a\|}{5\lambda(1+\delta)}$, then

$$\|f(a) - f(b)\| \geq \frac{\|b-a\|}{10\lambda^2(1+\delta)^2}.$$

Otherwise,

$$\begin{aligned} (3.6) \quad & \|f_{n+1}(b) - f_{n+1}(a)\| - (1 - \lambda_b)\|b\| - \lambda_a\|a\| \\ & \geq \frac{\|b-a\|}{\lambda(1+\delta)} - 2(\|b\| - 2^{n+1}) - 2(2^{n+1} - \|a\|) \\ & \geq \frac{\|b-a\|}{\lambda(1+\delta)} - \frac{4}{5} \frac{\|b-a\|}{\lambda(1+\delta)} \\ & \geq \frac{\|b-a\|}{5\lambda(1+\delta)} \end{aligned}$$

Finally,

$$\frac{\|b-a\|}{10\lambda^2(1+\delta)^2} \leq \|f(a) - f(b)\| \leq 9\|a-b\|.$$

$$(3) \quad 2^n \leq \|a\| < 2^{n+1} < 2^p \leq \|b\| < 2^{p+1}$$

We have

$$\begin{aligned} \|\Pi_n(f(a) - f(b))\| &= \|\lambda_a f_n(a)\|, \\ \|\Pi_{n+1}(f(a) - f(b))\| &= \|(1 - \lambda_a)f_{n+1}(a)\|, \\ \|\Pi_p(f(a) - f(b))\| &= \|\lambda_b f_p(b)\| \\ \text{and } \|\Pi_{p+1}(f(a) - f(b))\| &= \|(1 - \lambda_b)f_{p+1}(b)\|. \end{aligned}$$

Hence,

$$\|f(a) - f(b)\| \leq \|a\| + \|b\|$$

and using the projections Π_p and Π_{p+1} ,

$$\|f(a) - f(b)\| \geq \frac{1}{4} \left(\frac{\lambda_b \|b\|}{\lambda(1+\delta)} + \frac{(1 - \lambda_b)\|b\|}{\lambda(1+\delta)} \right) = \frac{\|b\|}{4\lambda(1+\delta)}.$$

But we have the following inequalities,

$$\frac{\|b\|}{2} \leq \|b\| - \|a\| \leq \|b-a\| \leq \|b\| + \|a\| \leq 2\|b\|.$$

And thus,

$$\frac{\|b-a\|}{8\lambda(1+\delta)} \leq \|f(a) - f(b)\| \leq 4\|a-b\|.$$

Finally we get the following global estimates:

$$\frac{\|b-a\|}{20\lambda^2(1+\delta)^2} \leq \|f(a) - f(b)\| \leq 9\|a-b\|.$$

□

Remark 3.4 We can reconstruct the proof of theorem 3.1 combining proposition 3.3 and the fact that $C([0, 1])$ is a \mathcal{L}_∞ -space and isometrically universal for all separable metric spaces.

Together for $p = 2$ our statement is theorem 3.2.

4. COARSE BI-LIPSCHITZ EMBEDDING OF PROPER METRIC SPACES

f is a coarse bi-Lipschitz embedding if there exists two non negative constants C_d and C_a such that for all $x, y \in M$,

$$\frac{1}{C_d}d(x, y) - C_a \leq \delta(f(x), f(y)) \leq C_d d(x, y) + C_a.$$

In the sequel we will use the notation $M \xrightarrow{(C_d, C_a)} N$. A coarse bi-Lipschitz embedding need not be injective neither continuous. The constant C_d will be called the *dilation coarse bi-Lipschitz constant*, and C_a the *additive coarse bi-Lipschitz constant*.

We prefer to use the term coarse bi-Lipschitz embedding instead of quasi-isometric embedding which is used for instance in [19], because in Nonlinear Functionnal Analysis the term quasi-isometric embedding refers to a Lipschitz embedding f such that $\text{dist}(f) \leq 1 + \epsilon$, for all $\epsilon > 0$. We remark that according to N. J. Kalton's terminology from [14], a coarse bi-Lipschitz embedding f is in particular a *coarse Lipschitz map*, which means that $\limsup_{t \rightarrow \infty} \frac{\omega_f(t)}{t} < \infty$.

Corollary 4.1. *Let $1 \leq p \leq \infty$ and $\lambda \geq 1$. Let $(Y, \|\cdot\|)$ a $\mathcal{L}_{p, \lambda}$ -space and M a proper subset of Y , then M admits a coarse bi-Lipschitz embedding into any Banach space X which contains uniformly the ℓ_p^n 's. Moreover the additive and dilation coarse bi-Lipschitz constants are universal, they depend only on λ .*

Proof : $B_n := \{t \in M; \|t\| \leq 2^{n+1}\}$ is compact. Let R_n an $\frac{\epsilon}{2}$ -net of B_n . The cardinal of R_n is finite. $R := \bigcup_n R_n$ is an $\frac{\epsilon}{2}$ -net of M , and is locally finite. Define

$$\begin{aligned} \beta : M &\rightarrow R \\ t &\mapsto \beta(t), \text{ such that } \|t - \beta(t)\| < \frac{\epsilon}{2} \end{aligned}$$

It is straightforward that $\|a - b\| - \epsilon \leq \|\beta(a) - \beta(b)\| \leq \|a - b\| + \epsilon$. If we embed R using theorem 3.3 and compose with β then we build a coarse bi-Lipschitz embedding with universal constants C_d and C_a .

□

Actually in the case $p = \infty$, G. Schechtman indicated to us an argument to prove that the converse statement holds. We thank him for allowing us to present it there.

Proposition 4.2. *The following assertions are equivalent :*

- i) X has no nontrivial cotype
- ii) There exists a universal constant $C_d > 0$ such that for every proper metric space (M, d) and for all $\epsilon > 0$, $M \xrightarrow{(C_d, \epsilon)} X$
- iii) There exist two universal positive constants C_d and C_a such that for every proper metric space (M, d) , $M \xrightarrow{(C_d, C_a)} X$

Proof : i) implies ii) is Corollary 4.1 and ii) implies iii) is trivial.

Let us prove iii) implies i) using G. Schechtman's argument.

Fix n and k be two positive integers, and let $N_k = \frac{1}{k}\mathbb{Z}^n \cap kB_{\ell_\infty^n}$. N_k is finite, hence there exists an embedding $\Theta_k : N_k \rightarrow X$ such that $\Theta_k(0) = 0$ and $N_k \xrightarrow{(C_d, C_a)} X$. Let $\Phi_k(x) = \frac{\Theta_k(kx)}{k}$, then

$$\frac{1}{C_d}\|x - y\|_\infty - \frac{C_a}{k} \leq \|\Phi_k(x) - \Phi_k(y)\| \leq C_d\|x - y\|_\infty + \frac{C_a}{k}.$$

Define Ψ_k from $B_{\ell_\infty^n}$ to N_k such that for all $x \in B_{\ell_\infty^n}$, we have

$$\|x - \Psi_k(x)\|_\infty \leq \frac{1}{k}.$$

$$\begin{aligned} \Phi : B_{\ell_\infty^n} &\rightarrow X_{\mathcal{U}} \\ x &\mapsto \overline{(\Phi_k \circ \Psi_k(x))_{k \geq 1}}^{\mathcal{U}}, \end{aligned}$$

where $X_{\mathcal{U}}$ is the ultra-product of X according to a nontrivial ultrafilter \mathcal{U} of \mathbb{N} . Φ is a Lipschitz embedding with distortion C_d the dilation coarse bi-Lipschitz constant. Using a delicate w^* -differentiability argument due to S. Heinrich and P. Mankiewicz [11], we can prove that ℓ_∞^n linearly embeds into $X_{\mathcal{U}}^{**}$. And the fact that the double-dual and the ultra-product of a Banach space Y are finitely representable in Y allows us to conclude. □

Remark 4.3 Using the argument of G. Schechtman one also can prove that the converse of theorem 2.1 in [5] holds. Namely, a Banach space X uniformly contains the ℓ_∞^n 's if and only if every locally finite metric space Lipschitz embeds into X .

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